Recall: $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$

ex: $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$ converges or diverges ?

$$\frac{1}{3!} + \frac{2!}{5!} + \frac{3!}{7!} + \cdots$$

$$L = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(2(n+1)+1)!}}{\frac{n!}{(2n+1)!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{(2n+1)!}{(2n+3)!}$$

$$= \lim_{n\to\infty} (n+1) \cdot \frac{1}{(2n+2)(2n+3)}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{4n^2+\dots} = 0$$

By ratio test, series converges absolutely.

ex: $\sum_{n=1}^{\infty} n^2$ is a divergent series

Let's apply ratio test.

$$L = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2}$$

$$= 1$$

No conclusion from ratio test

ex: $\sum_{n=1}^{\infty} \frac{e^n}{n^e}$

$$\sum_{n=1}^{\infty} \frac{e^n}{n^e} = \frac{e}{1^e} + \frac{e^2}{2^e} + \frac{e^3}{3^e} + \dots$$

$$L = \lim_{n\to\infty} \frac{\frac{e^{n+1}}{(n+1)^e}}{\frac{e^n}{n^e}}$$

$$= \lim_{n\to\infty} \frac{e^{n+1}}{e^n} \cdot \frac{n^e}{(n+1)^e}$$

$$= \lim_{n\to\infty} e^{n+1-n} \left(\frac{n}{n+1}\right)^e$$

$$= \lim_{n\to\infty} e \cdot \left(\frac{n}{n+1}\right)^e = e \cdot \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^e$$

$$= e \cdot 1^e \rightarrow e > 1$$

$L > 1 \Rightarrow$ series diverges

proof (ish) of ratio test

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \Rightarrow$ if n large $\frac{|a_{n+1}|}{|a_n|} \approx L$

$$|a_{n+1}| \approx |a_n| \cdot L$$

$$|a_{n+2}| \approx |a_{n+1}| \cdot L \approx |a_n| \cdot L^2$$

$$|a_{n+k}| \approx L^k |a_n|$$

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{n-1} |a_k| + \sum_{k=n}^{\infty} |a_k|$$

$$|a_n| + |a_{n+1}| + |a_{n+2}| + \dots$$

$$\approx |a_n| + L|a_n| + L^2|a_n| + \dots$$

$$\approx |a_n|(1 + L + L^2 + \dots)$$

$$\approx |a_n| \frac{1}{1-L} \quad \text{if} \quad 0 \leq L \leq 1$$

Root Test

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = L$$

if $0 \leq L \leq 1$ $\quad \sum a_n$ converges absolutely

$L > 1$ $\quad \sum a_n$ diverges

$L = 1$ $\quad$ no conclusion

proof (ish)

if n is large $\Rightarrow \sqrt[n]{|a_n|} \approx L$

$$|a_n| \approx L^n$$

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{n-1} |a_k| + \sum_{k=n}^{\infty} |a_k|$$

$$= |a_n| + |a_{n+1}| + \dots$$

$$\approx L^n + L^{n+1} + \dots$$

$$= L^n(1 + L + L^2 + \dots)$$

if $0 \leq L < 1 \Rightarrow$ series converges abs.

ex: $a_n = \begin{cases} n/2^n, & n = \text{odd} \\ 1/2^n & n = \text{even} \end{cases}$

Does $\sum a_n$ converges?

* Let's try ratio test.

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{2^{n+1}}}{n/2^n} \quad \text{if } n = \text{odd}.$$

$$= \frac{1}{2n} \quad \text{if } n = \text{odd}$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)/2^{n+1}}{1/2^n}$$

$$= \frac{n+1}{2} \quad \text{if } n = \text{even}$$

Thus, $\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} 1/2^n & n = \text{odd} \\ \frac{n+1}{2} & n = \text{even} \end{cases}$

$\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist. Can not apply ratio test.

* Let's try root test.

$$\lim_{n\to\infty} |a_n|^{1/n} = \begin{cases} \left(n/2^n\right)^{1/n}, & n = \text{odd} \\ \left(1/2^n\right)^{1/n}, & n = \text{even} \end{cases}$$

$$= \begin{cases} \frac{n^{1/n}}{2} & n = \text{odd} \\ 1/2 & n = \text{even} \end{cases}$$

Recall:

$$\lim_{n\to\infty} n^{1/n} = 1.$$

Thus,

$$\lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{2} = L \quad \text{(since converge)}$$

(book 40) **ex:** $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{n/2}}$ converges or diverges

$$L = \lim_{n\to\infty} \frac{n^{1/n}}{(\ln n)^{1/2}} = \frac{1}{+\infty} = 0$$

Converges absolutely by root test.

(book 42) **ex:** $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3 \cdot 2^n}$

$$L = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{3^n}{n^3 \cdot 2^n}\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{3}{n^{3/n} \cdot 2} = \frac{3}{2} \lim_{n\to\infty} \frac{1}{(n^{1/n})^3} \to \frac{3}{2} > 1$$

diverges.

10.6 ALTERNATING SERIES AND CONDITIONAL CONVERGENCE

A series in which the terms are alternately positive and negative is an <u>alternatig series</u>.

Theorem: Alternating Series Test

The series

$$\sum^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

Converges if all three conditions are satisfied:

1- $a_n \geq 0$ for all n

2- $a_n \geq a_{n+1}$ for all n

3- $\lim_{n\to\infty} a_n = 0$

ex: $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$a_n = \frac{1}{n}$, $n = 1, 2, 3.$

i) $a_n \geq 0$

ii) $a_n \geq a_{n+1}$ why? $\rightarrow \frac{1}{n} > \frac{1}{n+1}$

iii) $\lim_{n\to\infty} \frac{1}{n} = 0$

All conditions are satisfied $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by A.S.T.

NOTE: $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Harmonic series diverges

So $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but does not converges absolutely.

Definition: If a series converges but not converges absolutely then it is called conditionally convergent series.

ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

proof of A.S.T

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot a_n$$

$$S_{2m} = (a_1 - a_2) + (a_3 - a_4) + \dots + (a_{2m-1} - a_{2m}) \geq 0$$

$$S_{2m} \geq 0$$

$$S_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2m-2} - a_{2m-1}) - a$$

$$S_{2m} \leq 0$$

$$\leq a_1$$

$$S_{2m+2} - S_{2m} = a_{2m+1} - a_{2m+2} \geq 0$$

$$0 \leq S_{2m} \leq a_1$$

$$0 \leq S_{2m} \leq S_{2m+2} \leq a_1$$

(*) S_{2m} increases with m and bounded from.

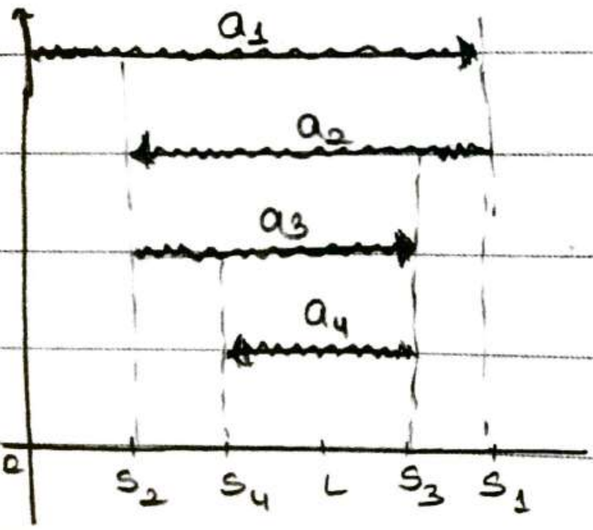
Hence,

$$\lim_{m \to \infty} S_{2m} = L$$

$$S_{2m+1} = S_{2m} + a_{2m+1}$$

$$\lim_{m \to \infty} S_{2m+1} = \lim_{m \to \infty} S_{2m} + \underbrace{\lim_{m \to \infty} a_{2m+1}}_{=0}$$

$$= L$$



$|L - S_1| \leq a_2$

$|L - S_2| \leq a_3$

$|L - S_3| \leq a_4$

$\vdots$

$|L - S_n| \leq a_{n+1}$

ex: $L = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by A.S.T

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$$

$$|L - S_4| \leq \frac{1}{5}$$

$$\left|L - \frac{7}{12}\right| \leq \frac{1}{5}$$

(book 15) ex: $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n}{n^3+1}$ convergent absolutely?

convergent conditional?

divergent?

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \cdot \frac{n}{n^3+1} \right| = \sum_{n=1}^{\infty} \frac{n}{n^3+1}$$

⊛ Limit comparison test

$$\lim_{n \to \infty} \frac{\frac{n}{n^3+1}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{n^3}{n^3+1} = 1 \rightarrow 0 < 1 < \infty$$

Since limit

$\sum \frac{1}{n^2}$ and $\sum \frac{n}{n^3+1}$ have same character

$\sum \frac{1}{n^2}$ → converges (p=2 series)

$\sum \frac{n}{n^3+1}$ → must converges

⇒ Hence series converges absolutely

ex: $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{\ln n}{n}$ absolute converges?

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} = \frac{\ln 1}{1} + \frac{\ln 2}{2} + \frac{\ln 3}{3} + \cdots$$

$$\geq \frac{\ln 1}{1} + \frac{\ln 2}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\right)}_{+\infty}$$

$\sum_{n=1}^{\infty} \frac{\ln n}{n} = +\infty$ diverges absolutely.

conditional converges ? (use A.S.T)

i) $\frac{\ln n}{n} \geq 0 \qquad n \geq 1$

ii) $a_{n+1} \leq a_n$?

$$f(x) = \frac{\ln x}{x}$$

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0 \Rightarrow x > e$$

This means $\frac{\ln x}{x}$ is decreasing if $x > e$

$a_4 < a_3$

$a_5 < a_4 \qquad a_{n+1} \leq a_n$

$1/n \geq 3$

L'hospital

iii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0$

$\sum_{n=3}^{\infty} (-1)^{n+1} \cdot \frac{\ln n}{n}$ satisfies the conditions of A.S.T hence converges.

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{\ln n}{n}$ converges

Thus the series converge conditionally

Recall:

- $\sum a_n$ is said converge absolutely if $\sum |a_n|$ converge.
- if $\sum a_n$ is converges but $\sum |a_n|$ diverges, $\sum a_n$ is said to converge conditionally.

ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ this series converge by A.S.T

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = L$$

$$2L = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \dots$$

$$= 2 - 1 + \frac{1}{3}(2-1) - \frac{1}{2} + \frac{1}{5}(2-1) - \frac{1}{4} + \dots$$

$$= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \dots$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

$$= L$$

! Main Message: You can not rearrange the terms in a conditionally convergent series.

However, you can rearrage terms in an absolutely convergent series

10.7 POWER SERIES

A power series about $x=0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots$$

A power series about $x=a$ is

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

center $\rightarrow$ a

ex:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \begin{cases} \text{converges}, & |x|<1 \\ \text{diverges}, & |x|\geq 1 \end{cases}$$

geometric series

center $\rightarrow$ a

ex:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

For which volues of x, does the series convergent?

$$a_n = (-1)^{n-1} \cdot \frac{x^n}{n}$$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}/n+1}{x^n/n}\right|$$

$$= \lim_{n\to\infty} |x| \cdot \left|\frac{n}{n+1}\right| = |x|$$

$0 \leq L = |x| < 1 \Rightarrow$ series convergent abs.

$L = |x| > 1 \Rightarrow$ series divergent

③ what about when $|x| = 1$?

$$x = 1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$$ converges by A.S.T (but diverges absolutely)

$$x = -1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{1}{n} = -\infty$$

✓ diverges.

conclusion

series converge for $-1 < x \leq 1$

ex: $$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Same question?

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{2(n+1)-1} / 2(n+1)-1}{x^{2n-1} / 2n-1} \right|$$

$$= |x|^2 \cdot \lim_{n \to \infty} \frac{2n-1}{2n+1} = |x|^2$$

The series ✓ converges absolutely

if $|x|^2 < 1 \Leftrightarrow |x| < 1$

✓ diverges

if $|x| > 1$

what about when $|x| = 1$?

$x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

use A.S.T $\quad a_n = \frac{1}{2n-1} \to 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$

i) $a_n \geq 0$

ii) a_n's decreasing

iii) $\lim_{n\to\infty} a_n = 0$

The series converges by A.S.T

$x = -1 \to \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot (-1)^{2n-1}}{2n-1}$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n}}{2n-1}$ series converges by A.S.T

conclusion

series converge for $-1 \leq x \leq 1$

ex: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Same question

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{x^{n+1}/(n+1)!}{n! \cdot |x|^n}$

$= \lim_{n\to\infty} (n+1)|x| = \begin{cases} 0 & |x| = 0 \\ +\infty & |x| \neq 0 \end{cases}$

NOTE: A power series always converge at its center.

THEOREM: $\sum c_n (x-a)^n$

1. Either there exists $R > 0$ such that the series converges absolutely for $|x-a| < R$, diverges for $|x-a| > R$. The series may not converge at the end points $x = a-R$, $x = a+R$

2. The series converges absolutely for every x. $(R = +\infty)$

3. The series converges at $x = a$ and diverges else where $(R = 0)$

R = radius of convergence. The interval at which the power series converges is called interval of convergence.

ex: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ interval of convergence $\Rightarrow (-\infty, +\infty)$

center = a

$R = +\infty$

ex: $\sum_{n=0}^{\infty} n! \cdot x^n$ interval of convergence $= \{0\}$

$R = 0$

ex: $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^{2n-1}}{2n-1}$ interval of convergence $= [-1, 1]$

$R = 1$

center $= 0$

Theorem: If

$\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence R

then $f(x)$ is differentiable

on $(a-R, a+R)$ and

- $f'(x) = \sum_{n=1}^{\infty} c_n \cdot n (x-a)^{n-1}$ converges for $x \in (a-R ; a+R)$
- $f''(x) = \sum_{n=2}^{\infty} c_n\, n(n-1)(x-a)^{n-2}$ converges for $x \in (a-R, a+R)$

ex: $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$.

$1 + x + x^2 + x^3 + \dots$

$$f'(x) = \frac{d}{dx} (1-x)^{-1} = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n \cdot x^{n-1} = 1 + 2x + 3x^2 + \dots$$

for $|x| < 1$

$$f''(x) = \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)\, x^{n-2} \qquad |x| < 1$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) \cdot x^n$$

ex: Find the value of the sum

$$1 + 2x + 3x^2 + 4x^3 + \dots$$

$$x = 0.49 \quad ?$$

for

$$\sum_{n=1}^{\infty} n \cdot x^{n-1} = \frac{1}{(1-x)^2}, \quad |x| < 1$$

$$\sum_{n=1}^{\infty} n\,(0.49)^{n-1} = \frac{1}{(0.51)^2}$$

Theorem:

If $f(x) = \sum_{n=0}^{\infty} c_n \cdot (x-a)^n$ has radius of convergence $R > 0$

Then,

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \cdot \frac{(x-a)^{n+1}}{n+1} + c \quad \text{converges for} \quad a-R < x < a+R$$

ex: Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} \dots \quad |x| < 1$$

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)\, x^{2n}}{(2n+1)} = \underbrace{1 - x^2 + x^4 - x^6 + \dots}_{|x| < 1}$$

$$= \sum_{n=0}^{\infty} (-x^2)^n, \quad |x| < 1$$

$$= \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}, \quad |x| < 1$$

$$\int f'(x)\,dx = \int \frac{1}{1+x^2} \cdot dx \rightsquigarrow f(x) = \arctan x + c$$

$f(x) = \arctan x + C$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)} = \arctan x + C$$

$x = 0 \Rightarrow \quad 0 = \arctan 0 + C$

$0 = 0 + C$

$\hookrightarrow C = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots = \arctan x \quad , |x| < 1$$